Functions

Part Two

Outline for Today

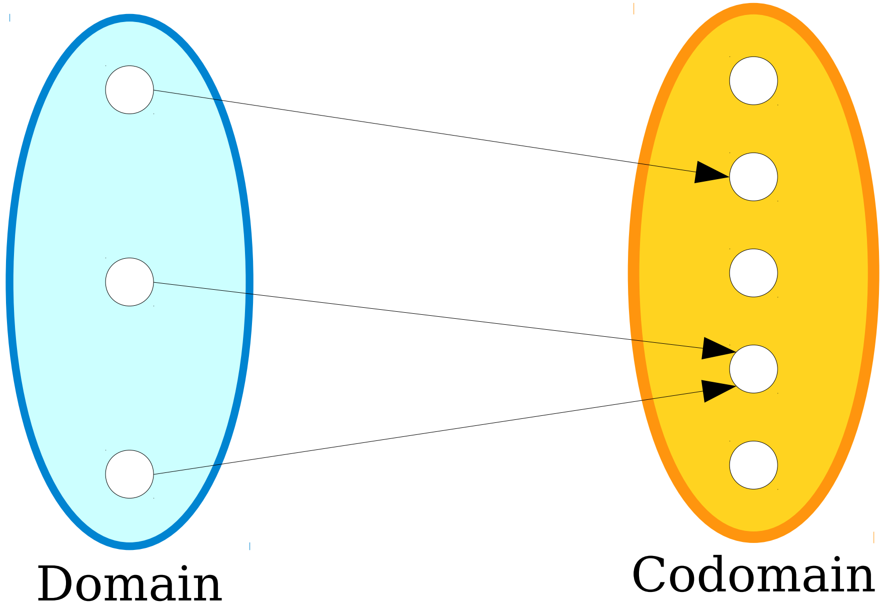
- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it's relevant. 😀
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.
- ***Function Composition***
  - Sequencing functions together.

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, $f(x)$ belongs to the codomain.
- We write $\boldsymbol{f : A \to B}$ to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.

The function must be defined for each element of its domain.



The output of the function must always be in the codomain, but not all elements of the codomain need to be covered.

Involutions

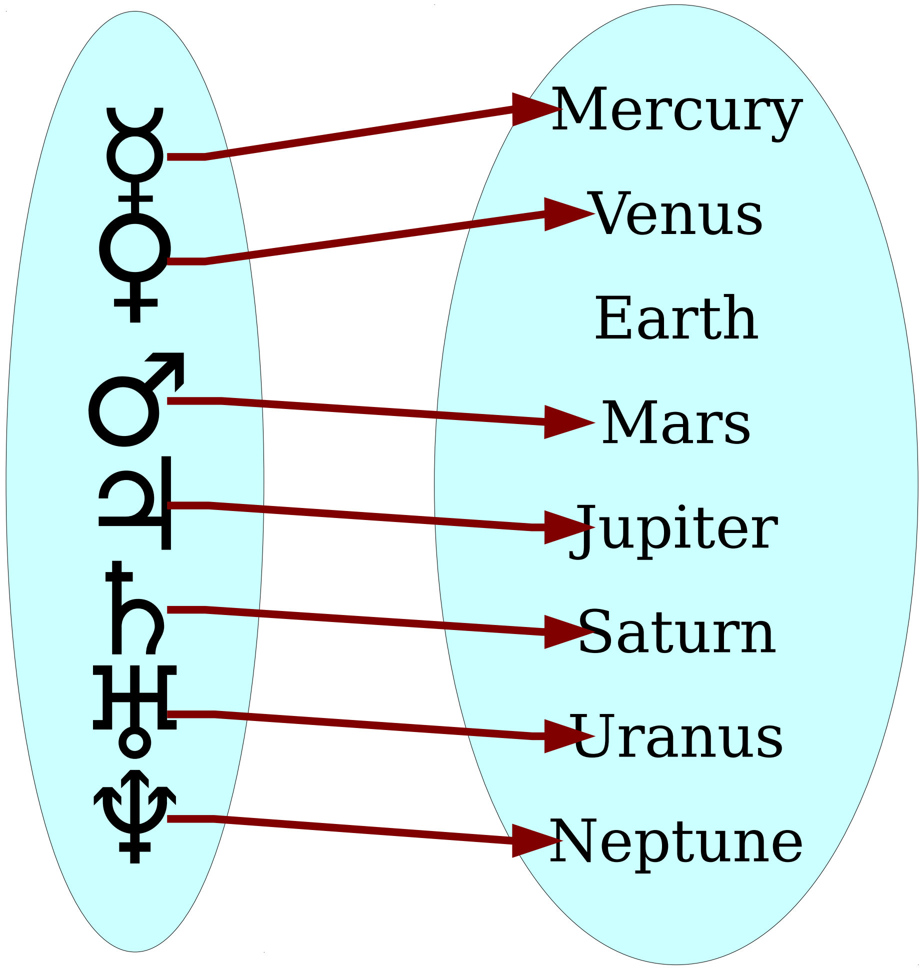
- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

Injective Functions

- $\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$
- $\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$



Review: Injective Functions

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

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What does it mean for the function f to be injective?

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What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ f(n_1) = f(n_2) \to n_1 = n_2\ )$$

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ n_1 \neq n_2 \to f(n_1) \neq f(n_2)\ )$$

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Your Turn: Write the Assume and WTS steps of a Direct Proof approach to this proof, for the **first** definition of injective. *(Remember that direct proof is for proving theorems that are implications—in this case that implication is in the definition of injectivity.)*

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Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required. ■

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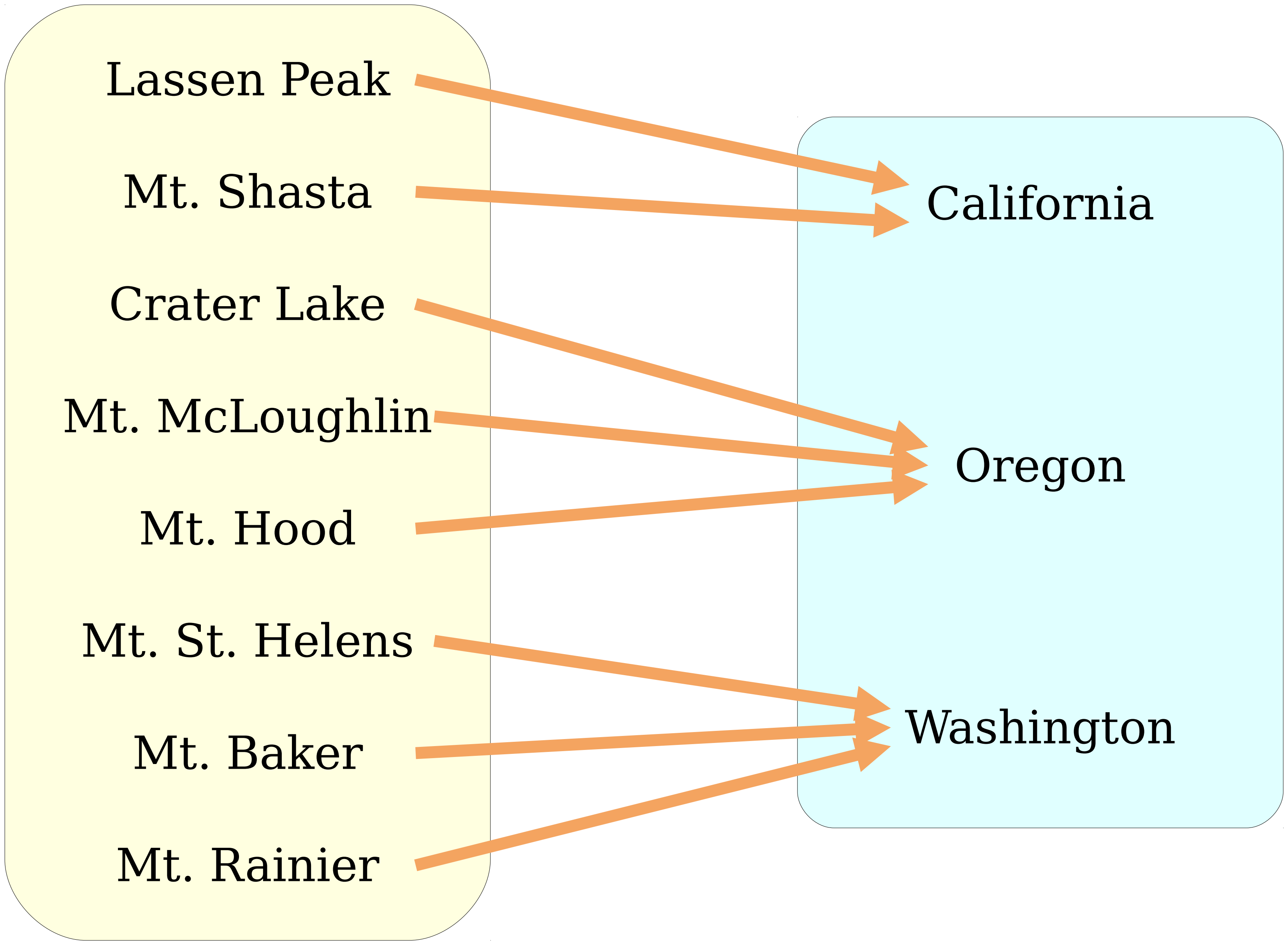
Your Turn: Write the Assume and WTS steps of a Direct Proof approach to this proof, for the **2nd** definition of injective. *(Remember that direct proof is for proving theorems that are implications—in this case that implication is in the definition of injectivity.)*

New: Another Class of Functions


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*“For every output, there's an input that produces it.”*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

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What does it mean for *f* to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Your Turn: Write the Assume and WTS steps for this proof, using the definition of surjective.

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$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

Pop Quiz!
Which row(s) of this proof techniques table did we use for for that proof?

A Proof About Birds



Theorem: If all birds can fly,
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$Bird(b)$, which says b is a bird;
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$CanFly(x)$, which says x can fly,

translate the theorem into first-order logic.

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Consider an arbitrary bird b. Since b is a bird, b can fly. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

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We never introduce a variable b.

We introduce a variable h almost immediately.

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds can fly.
  - Here, we ***proved*** all herons can fly.
- Statements behave differently based on whether you're assuming or proving them.

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Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

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Proving vs. Assuming

- If we ***assume*** the statement

$$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.

- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, can fly.

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	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	
$A \land B$	Prove A. Then prove B.	
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$.	
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	
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$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
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$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$.	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- We've seen three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{f \text{ is an involution.}} \quad \to \quad \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{f \text{ is surjective.}}$$

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Prove this.

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Assume this. Prove this.

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Ass

Prove this.

We've said that we need to ***prove*** this statement. How do we do that?

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$$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

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There's a universal quantifier up front. Since we're ***proving*** this, we'll pick an arbitrary $b \in A$.

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Now, we hit an existential quantifier. Since we're ***proving*** this, we need to find a choice of $a \in A$ where this is true.

Prove this.

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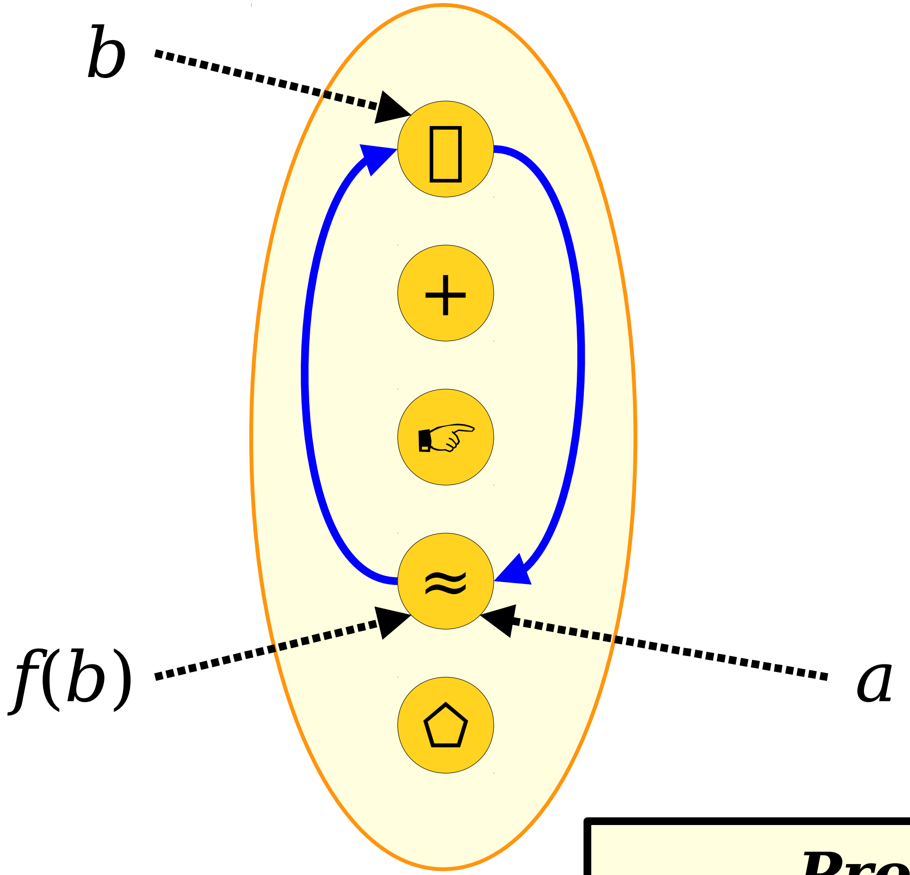

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Specifically, pick $a = f(b)$.

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We need to ***prove*** this part.
What does that mean?

Prove this.

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We now need to ***prove*** this implication. But we know how to do that! We assume the antecedent and prove the consequent.

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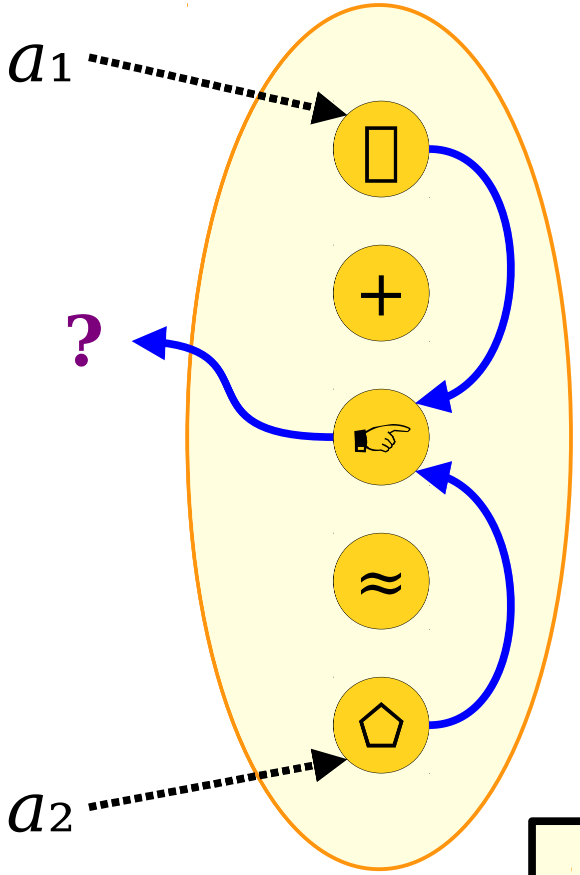

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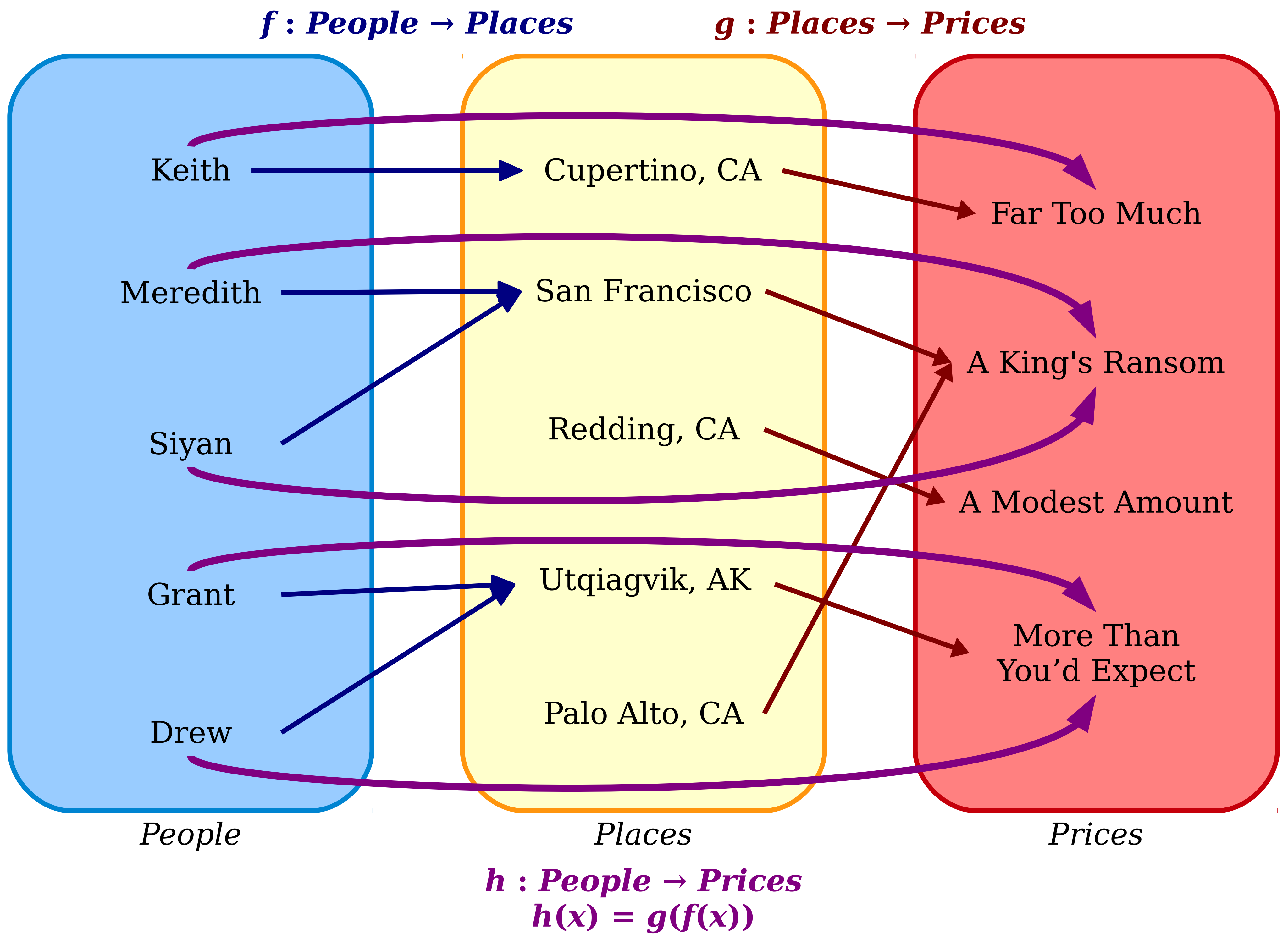
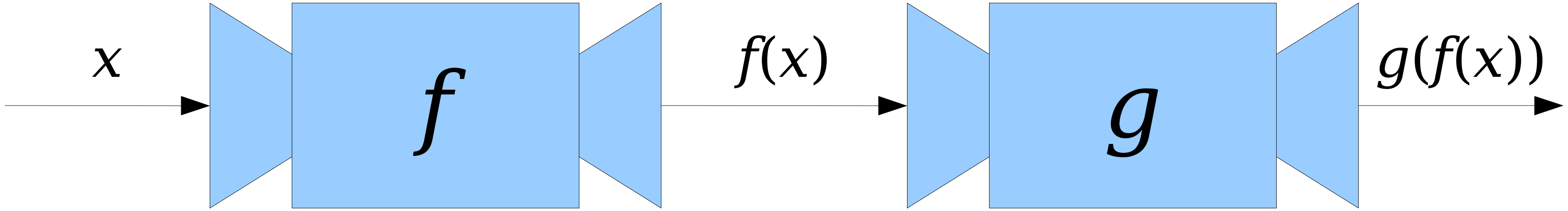
Function Composition


f : People → Places
g : Places → Prices
Keith
Meredith
Siyan
Grant
Drew
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than You’d Expect
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

> The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

Organizing Our Thoughts

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	

We need to ***prove*** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

We're ***assuming*** these universally-quantified statements, so we won't introduce any variables for what's here.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection. $\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$ $g : B \to C$ is an injection. $\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$ $a_1 \in A$ is arbitrarily-chosen. $a_2 \in A$ is arbitrarily-chosen.	$g \circ f$ is an injection. $\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$ We need to ***prove*** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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$f : A \to B$ is an injection.	$g \circ f$ is an injection.
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$g : B \to C$ is an injection.	
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

Now we're looking at an implication. Let's ***assume*** the antecedent and ***prove*** the consequent.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
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$g : B \to C$ is an injection.	Let's write this out separately and simplify things a bit.
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
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Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$
$f(x) \neq f(y)$
$)$

$g : B \to C$ is an injection.

$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$
$g(x) \neq g(y)$
$)$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$
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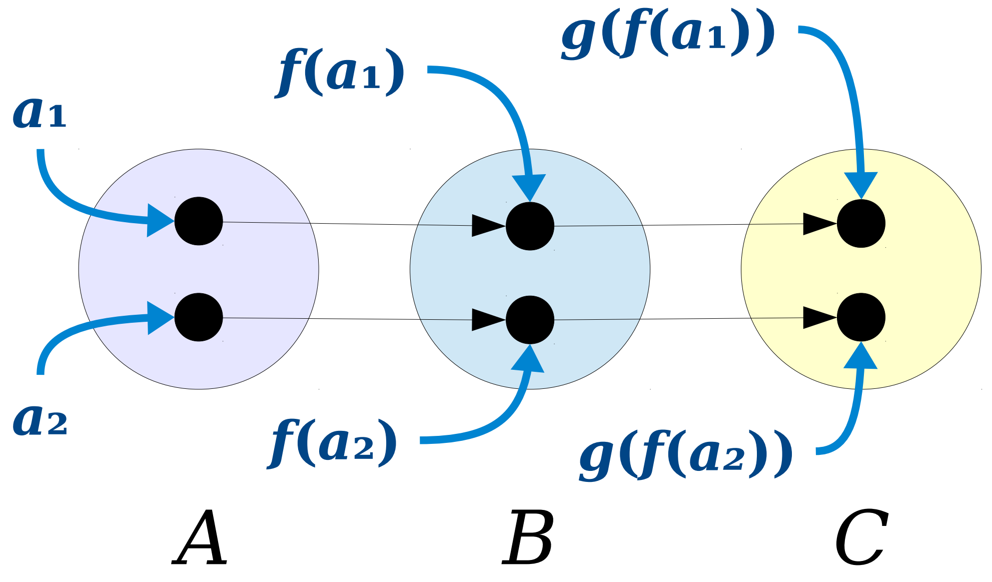
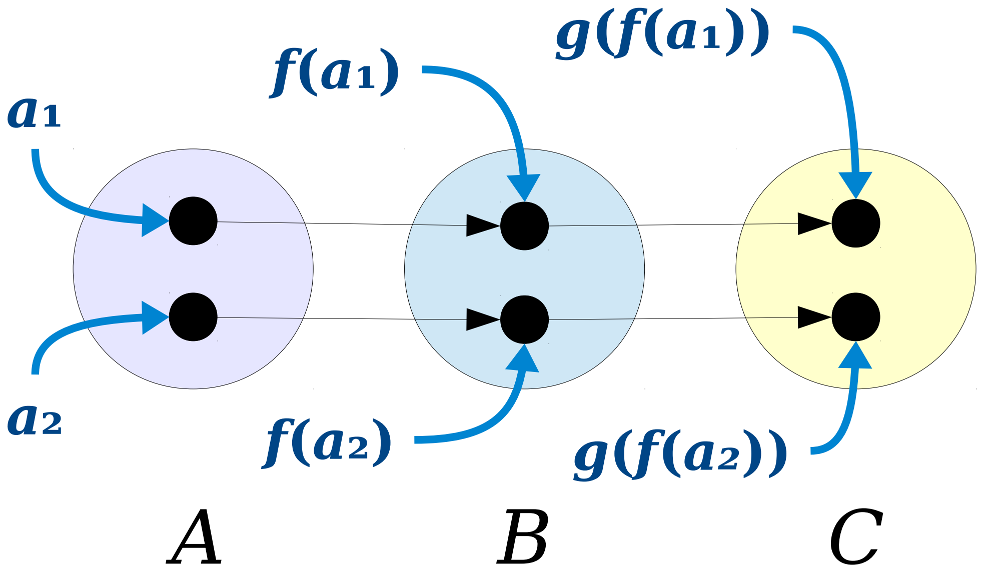
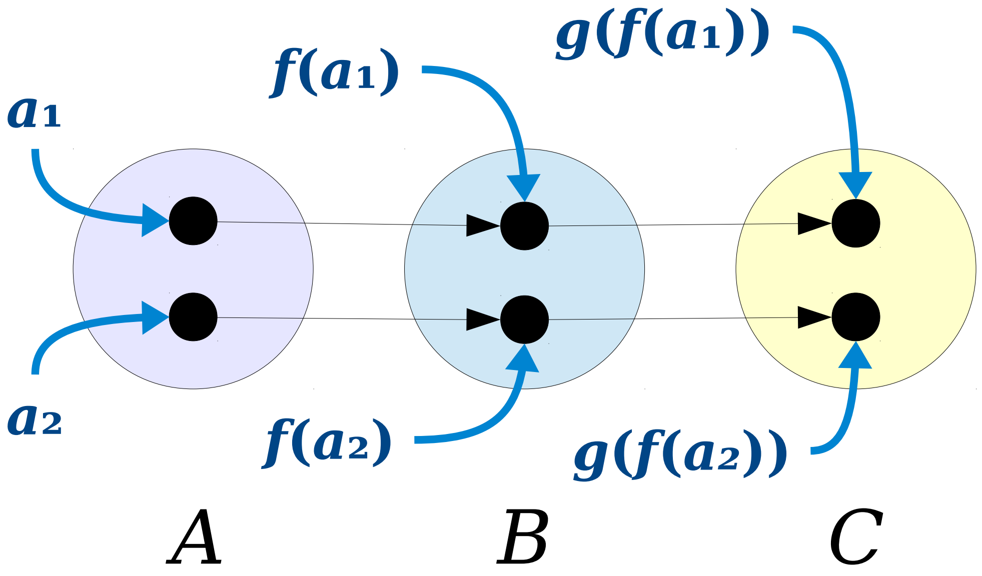


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.



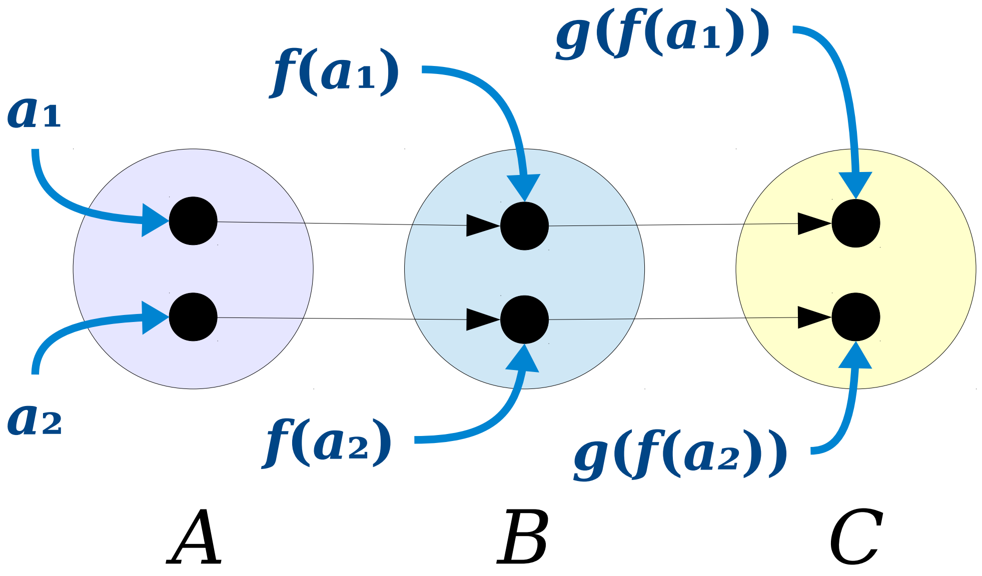
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof:



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections.



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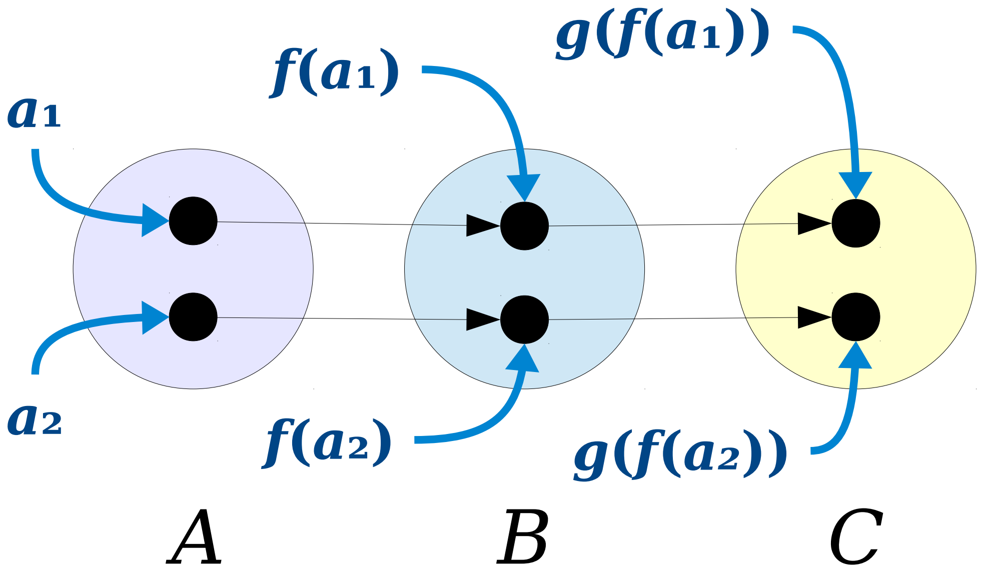
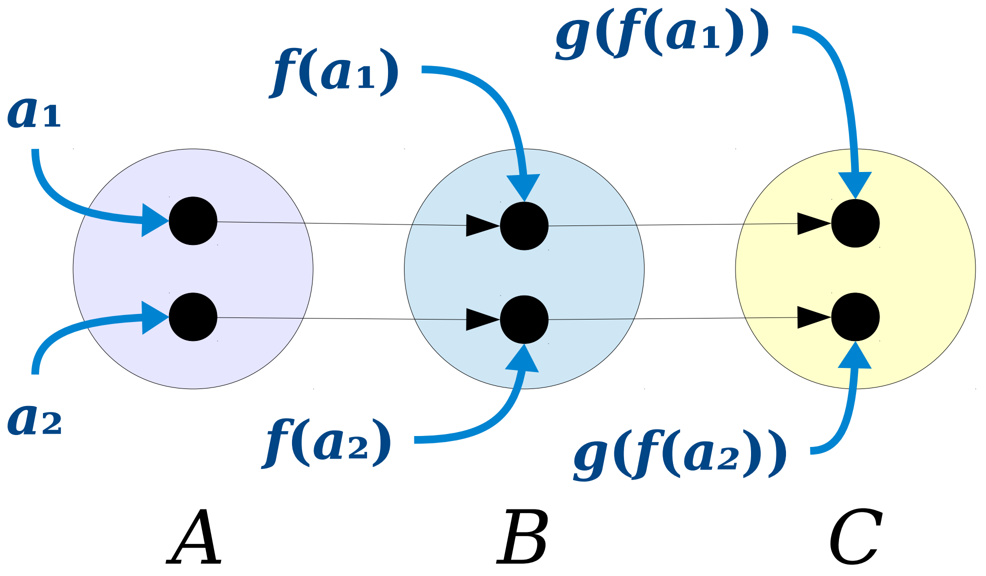
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$.



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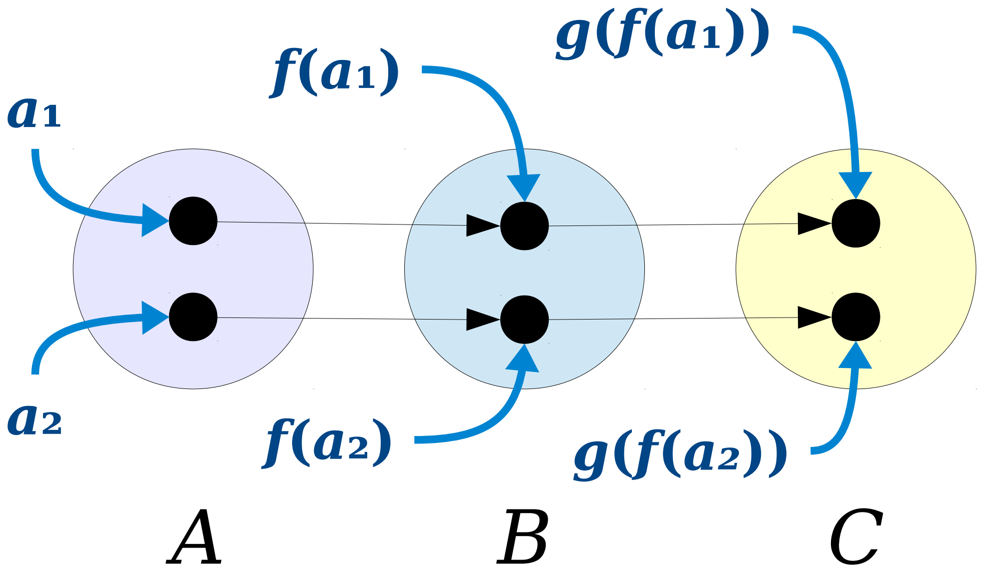

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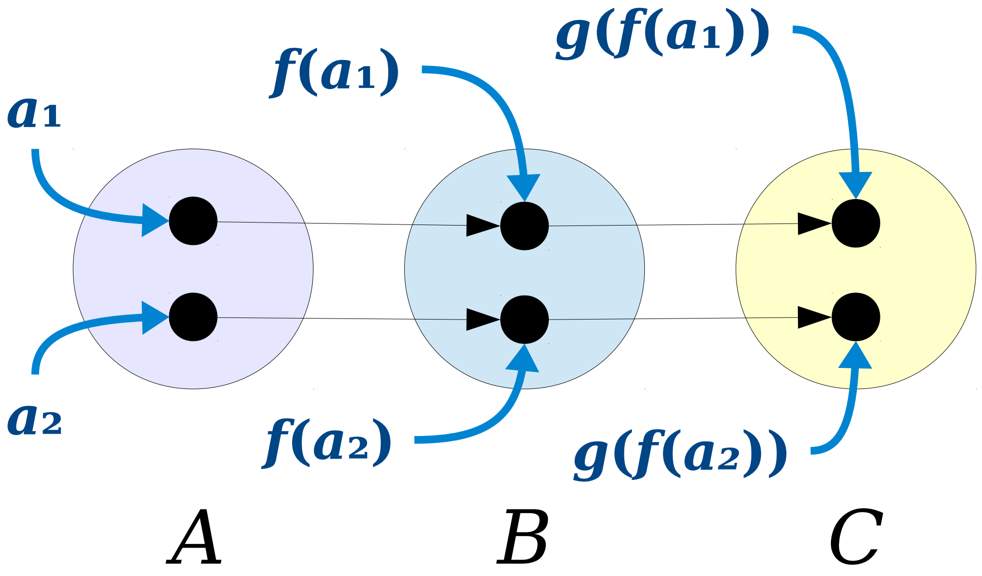

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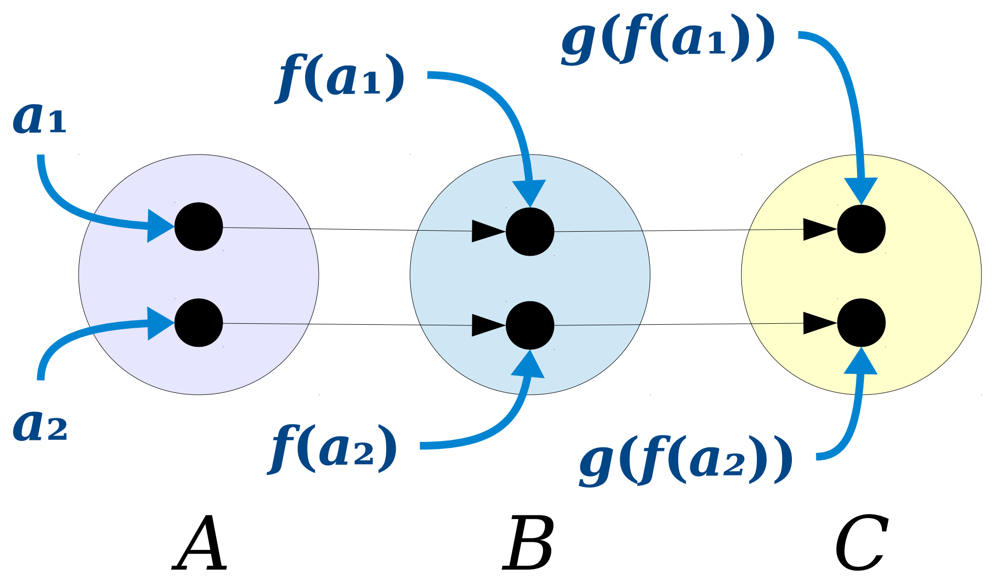
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.



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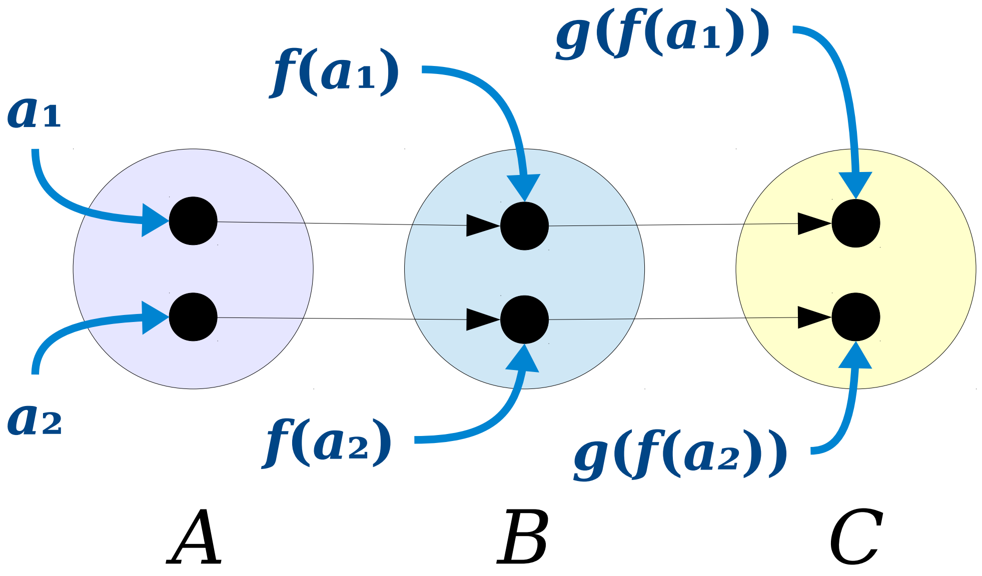
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required.



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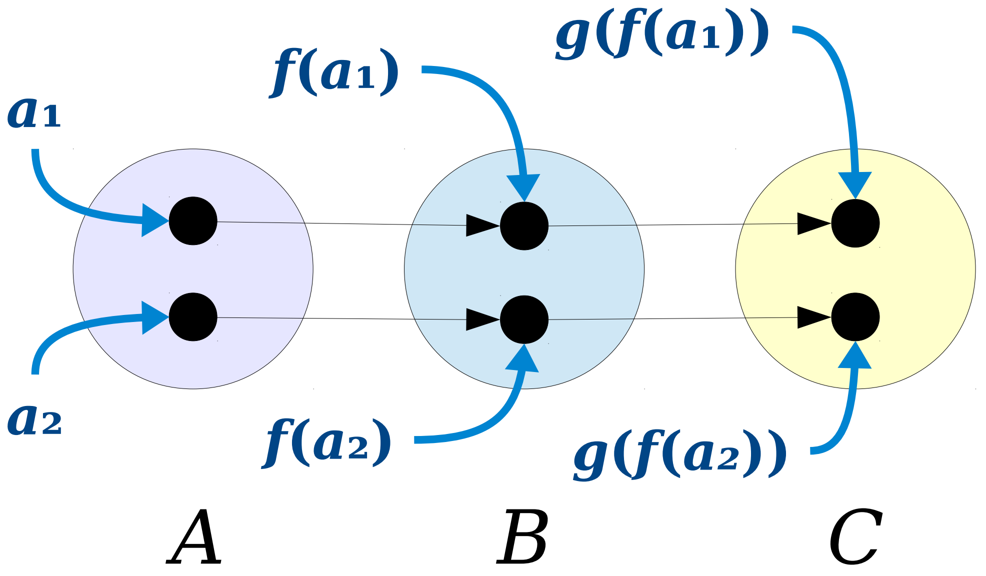

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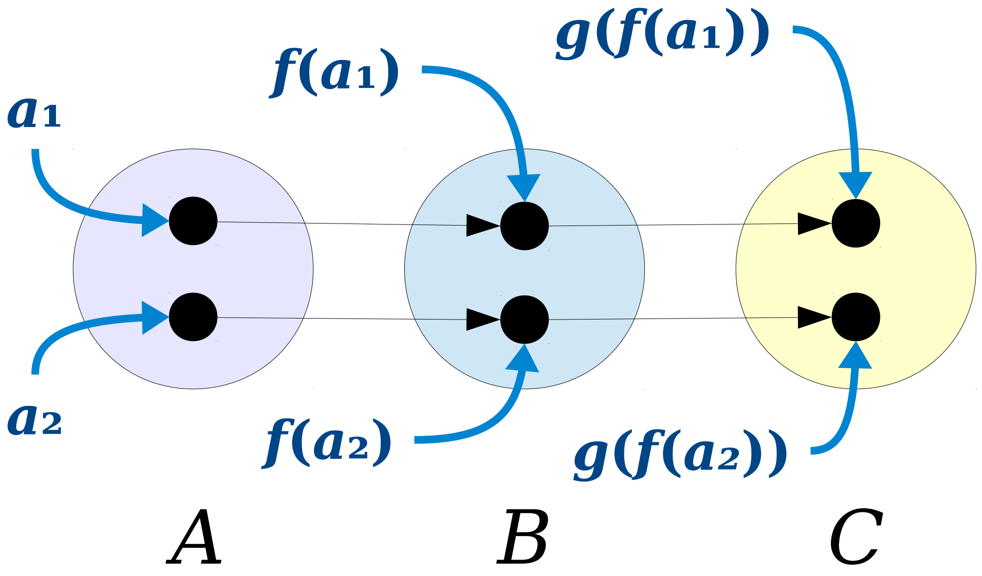
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.



Major Ideas From Today

- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.
- As always: try concrete examples, draw pictures, etc. before you dive into writing a proof.

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$.	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Cardinality Revisited***
  - Formalizing our definitions.
- ***The Nature of Infinity***
  - Infinity is more interesting than it looks!
- ***Cantor's Theorem Revisited***
  - Formally proving a major result.

Extra Slides

(The following is a proof of a theorem just like the one we just did with injection, but with surjection.)

Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof:

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Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections.

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What does it mean for $g \circ f : A \to C$ to be surjective?

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$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

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Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

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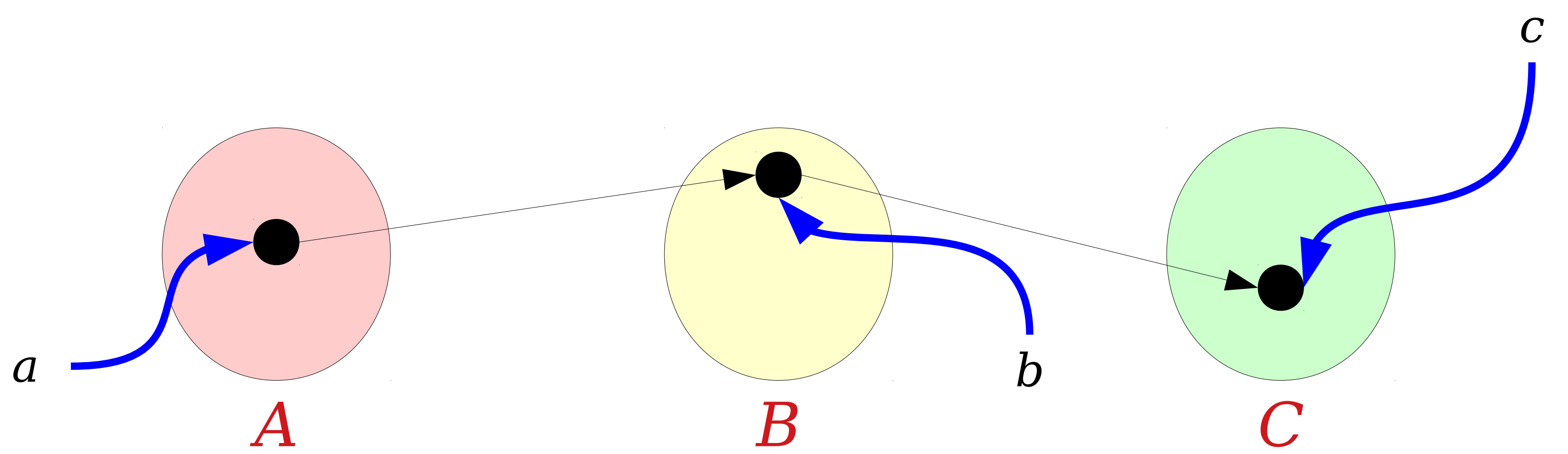

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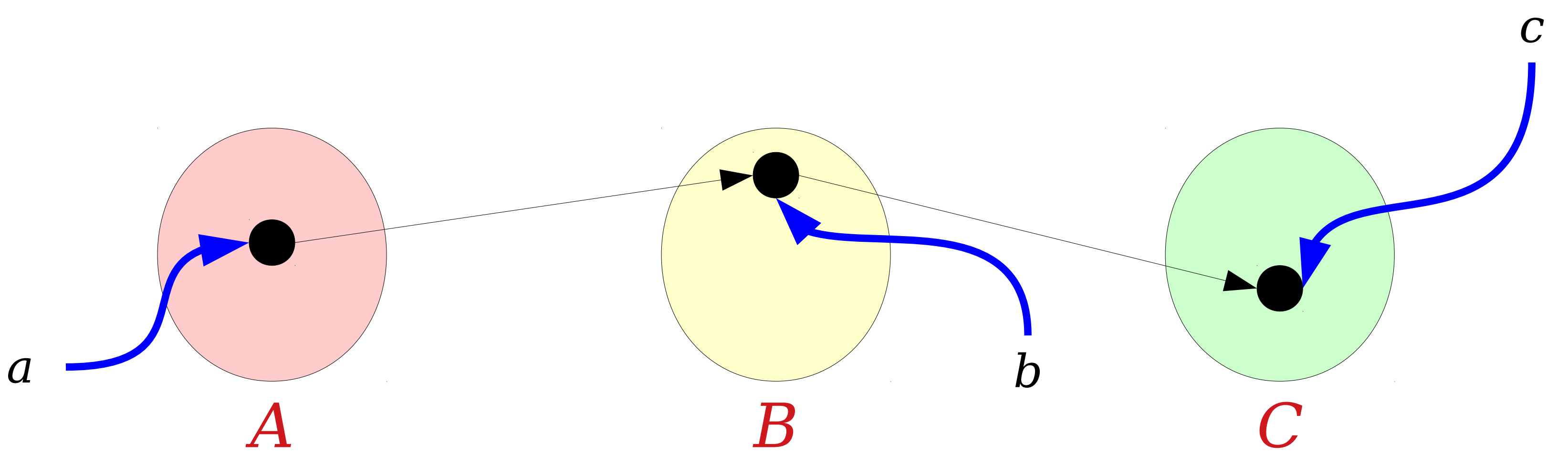
Consider any $c \in C$.



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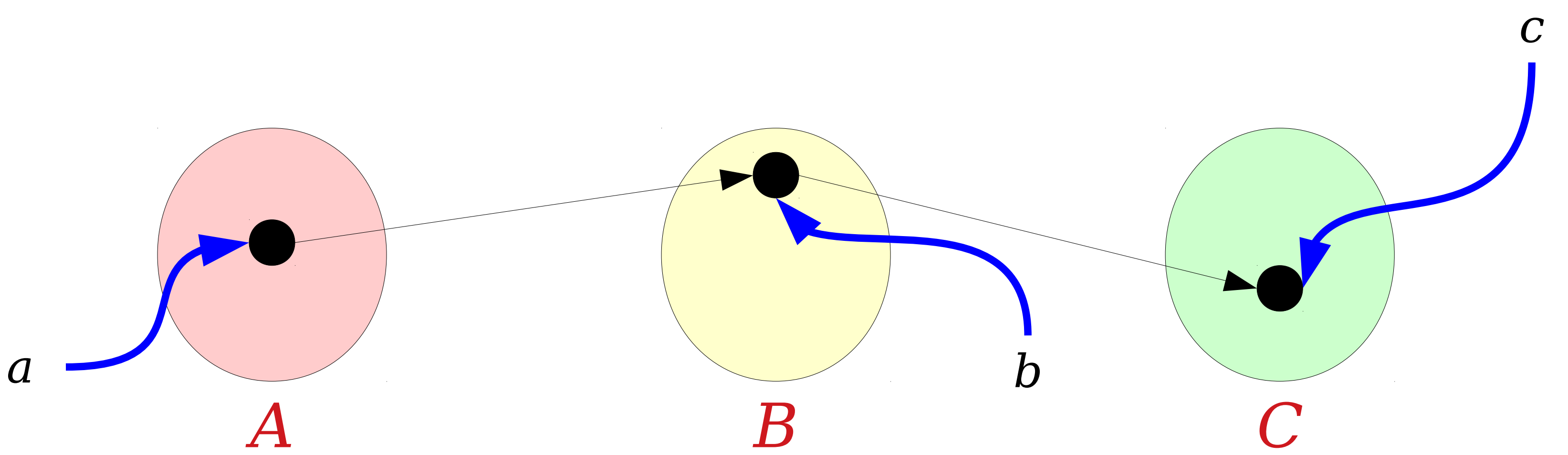
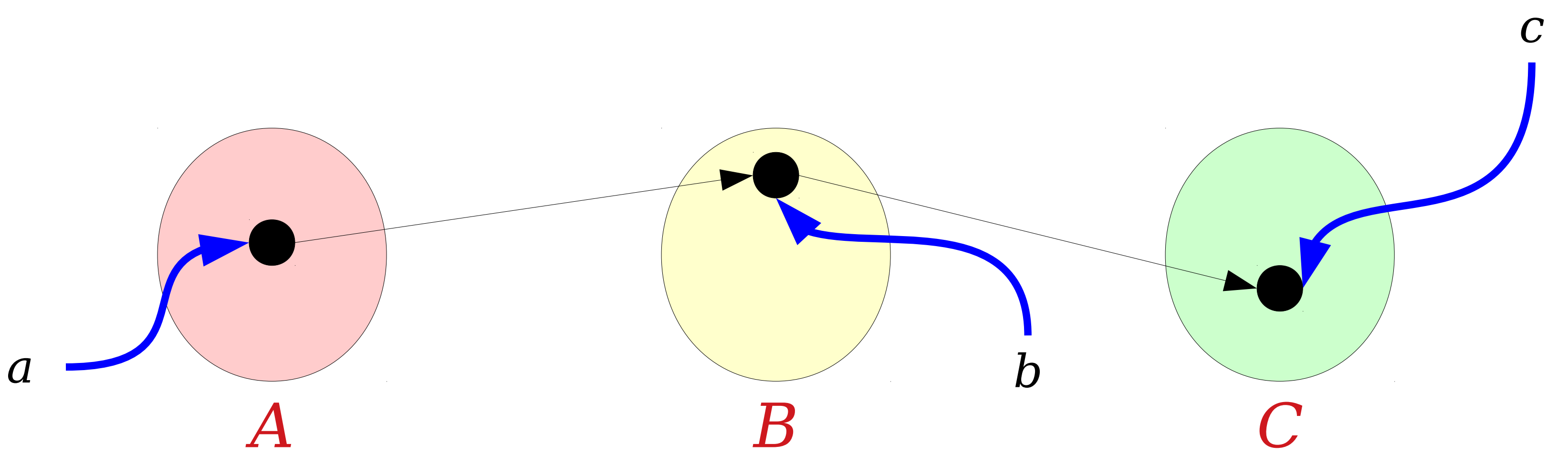
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$.



Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$.



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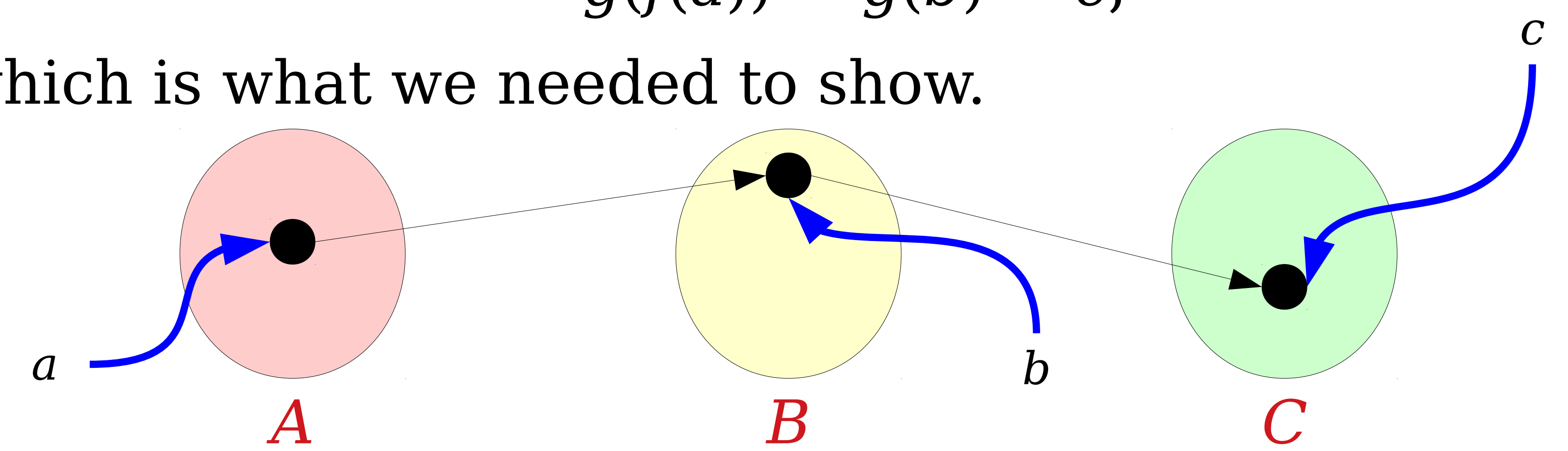
$$g(f(a)) = g(b) = c,$$

which is what we needed to show.



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